J. Indian Acad. Math.
Vol. 20, No. 2 (1998)

Rajeev P. Shrivastava

APPLICATION OF *I*-FUNCTION OF TWO VARIABLES IN PROBLEM OF VIBRATION IN A STRING

In this paper, we employ *I*-function of two variables defined by Goyal and Agrawal [2], to obtain a solution of the partial differential equation,

$$\frac{\partial^2 y}{\partial t^2} = \mu^2 \frac{\partial^2 y}{\partial x^2}, \; t > 0$$

related to a problem of vibration in a string.

1. Introduction

The object of this paper is to employ an integral involving *I*-function of two variables defined by Goyal and Agrawal [2], which is extension of *I*-function of one variable by Saxena V. P. [4], to obtain a solution of a problem of vibration in a string.

We consider the problem of vibration in a string to find the transverse displacement $y(x, t)$ in a string of length L stretched between the points (0, 0) and $(L, 0)$, if it is displaced initially into a position $y = f(x)$ and released from rest at this position with no external forces acting.

The required function $y(x, t)$ is the solution of the following boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = \mu^2 \frac{\partial^2 y}{\partial x^2}, \; t > 0, \; 0 \le x \le L, \tag{1.1}$$

and the initial conditions,

$$\left.\begin{aligned} & y(0, t) = 0;\ y(L, t) = 0 \\ & y(x, 0) = f(x),\ 0 \le x \le L \\ & \frac{\partial}{\partial t} y(x, 0) = 0,\ 0 \le x \le L \end{aligned}\right\}, \tag{1.1.1}$$

Therefore the general solution of the partial differential equation (1.1) is given by

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{\lambda_n \pi x}{L} \cos \frac{\lambda_n \pi \mu t}{L}, \tag{1.2}$$

Let us consider

$$y(x, 0) = \left(\sin \frac{\pi x}{L} \right)^{\omega - 1} I \left[\begin{matrix} z_1 \left(\sin \frac{\pi x}{L} \right)^{2k_1} \\ z_2 \left(\sin \frac{\pi x}{L} \right)^{2k_2} \end{matrix} \right], \tag{1.3}$$

where the I-function of two variables introduced by Goyal and Agrawal [2] in the following manner :

$$I^{m_1, n_1 : m_2, n_2 : m_3, n_3}_{p, q : p_i^{(1)}, q_i^{(1)} : p_i^{(2)}, q_i^{(2)} : r} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} [(e_p : E_p, E'_p)] : [(a_j, \alpha_j)_{1, n_2}], \\ [(f_q : F_q, F'_q)] : [(b_j, \beta_j)_{1, m_2}], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a_{ji}, \alpha_{ji})_{n_2 + 1, p_i^{(1)}}] : [(c_j, \gamma_j)_{1, n_3}], [(c_{ji}, \gamma_{ji})_{n_3 + 1, p_i^{(2)}}] \\ [(b_{ji}, \beta_{ji})_{m_2 + 1, q_i^{(1)}}]; [(d_j, \delta_j)_{1, m_3}], [(d_{ji}, \delta_{ji})_{m_3 + 1, q_i^{(2)}}] \end{matrix} \right]$$

$$= - \frac{1}{4\pi^2} \int_{L_1} \int_{L_2} \varphi_1(\xi) \varphi_2(\eta) \psi(\xi, \eta)\, z_1^{\xi} z_2^{\eta}\, d\xi\, d\eta, \tag{1.4}$$

where

$$\Phi_1(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n_2} \Gamma(1 - a_j + \alpha_j \xi)}{\sum_{i=1}^{r} \left[\prod_{j=m_2+1}^{q_i^{(1)}} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \prod_{j=n_2+1}^{p_i^{(1)}} \Gamma(a_{ji} - \alpha_{ji} \xi) \right]} \tag{1.4.1}$$

$$\Phi_2(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(d_j - \delta_j \eta) \prod_{j=1}^{n_3} \Gamma(1 - c_j + \gamma_j \eta)}{\sum_{i=1}^{r'} \left[\prod_{j=m_3+1}^{q_i^{(2)}} \Gamma(1 - d_{ji} + \delta_{ji} \eta) \prod_{j=n_3+1}^{p_i^{(2)}} \Gamma(c_{ji} - \gamma_{ji} \eta) \right]} \tag{1.4.2}$$

$$\psi(\xi, \eta) = \frac{\prod_{j=1}^{m_1} \Gamma(f_j - F_j \xi - F'_j \eta) \prod_{j=1}^{n_1} \Gamma(1 - e_j + E_j \xi + E'_j \eta)}{\prod_{j=m_1+1}^{q} \Gamma(1 - f_i + F_i \xi + F'_j \eta) \prod_{j=n_1+1}^{p} \Gamma(e_i - E_j \xi - E'_j \eta)} \tag{1.4.3}$$

The double integral in (1.1), converges absolutely if

$$|\arg z_1| < \frac{A\pi}{2}, \quad |\arg z_2| < \frac{B\pi}{2}$$

where

$$A = \sum_1^{n_1} E_j - \sum_{n_1+1}^{p} E_j + \sum_1^{m_1} F_j - \sum_{m_1+1}^{q} F_j + \sum_1^{m_2} \beta_j - \sum_{m_2+1}^{q_i^{(1)}} \beta_{ji} + \sum_1^{n_2} \alpha_j - \sum_{n_2+1}^{p_i^{(1)}} \alpha_{ji} > 0, \tag{1.4.4}$$

$$B = \sum_1^{n_1} E'_j - \sum_{n_1+1}^{p} E'_j + \sum_1^{m_1} F'_j - \sum_{m_1+1}^{q} F'_j + \sum_1^{m_3} \delta_j - \sum_{m_3+1}^{q_i^{(2)}} \delta_{ji} + \sum_1^{n_3} \gamma_j - \sum_{m_3+1}^{p_i^{(2)}} \gamma_{ji} > 0, \tag{1.4.5}$$

we shall require the modified form of the integral [3, p. 372, Eq. (1)].

$$\int_0^L \left(\sin \frac{\pi x}{L} \right)^{\omega - 1} \sin \frac{\lambda_m \pi x}{L} \, dx = \frac{L \sin\left(\frac{\lambda_m \pi}{2} \right) \Gamma \omega}{2^{\omega - 1} \Gamma\left(\frac{\omega + \lambda_m + 1}{2} \right) \left(\frac{\omega - \lambda_m + 1}{2} \right)} \tag{1.5}$$

$$\operatorname{Re}(\omega) > 0$$

and the following orthogonal property [5, p. 28];

$$\int_0^L \sin\frac{\lambda_n \pi x}{L}\sin\frac{\lambda_m \pi x}{L}\,dx = \begin{cases} 0 & ; m \neq n \\ \frac{L}{2}\left(1 - \frac{1}{2\lambda_n \pi}\sin 2\lambda_n \pi\right) & ; m = n \end{cases} \tag{1.6}$$

Legendre's duplication formula,

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z + \frac{1}{2}\right). \tag{1.7}$$

and

$$p = m_2, n_2; m_3, n_3, Q = p_i^{(1)}, q_i^{(1)}; p_i^{(2)}, q_i^{(2)}; r \tag{1.8}$$

$$T = \left[\left(a_j, \alpha_j\right)_{1, n_2}\right], \left[\left(a_{ji}, \alpha_{ji}\right)_{n_2+1, p_i^{(1)}}\right]; \left[\left(c_j, \gamma_j\right)_{1, n_3}\right], \left[\left(c_{ji}, \gamma_{ji}\right)_{n_3+1, p_i^{(2)}}\right] \tag{1.9}$$

$$T' = \left[\left(b_j, \beta_j\right)_{1, m_2}\right], \left[\left(b_{ji}, \beta_{ji}\right)_{m_2+1, q_i^{(1)}}\right]; \left[\left(d_j, \delta_j\right)_{1, m_3}\right], \left[\left(d_{ji}, \delta_{ji}\right)_{m_3+1, q_i^{(2)}}\right]. \tag{1.10}$$

Throughout the paper, we use the notations P, Q, T, T' defined as per equations (1.8) to (1.10) respectively.

2. Integral

The following integral has been established in the paper :

$$\int_0^L = \left(\sin\frac{\pi x}{L}\right)^{m-1} \sin\frac{\lambda_n \pi x}{L}\; I\left[\begin{matrix} z_1\left(\sin\frac{\pi x}{L}\right)^{2k_1} \\ z_2\left(\sin\frac{\pi x}{L}\right)^{2k_2} \end{matrix}\right] dx$$

$$= \frac{L}{\sqrt{\pi}}\, I^{m_1, n_1+2:P}_{p+2, q+2:Q}\left[\begin{array}{c|l} z_1 & \left(1-\frac{\omega}{2};k_1,k_2\right), \left(\frac{1-\omega}{2};k_1,k_2\right), \left(e_p:E_p, E'_p\right):T \\ z_2 & \left(f_q;F_q,F'_q\right), \quad \left(\frac{1-\omega\pm\lambda_n}{2}:k_1,k_2\right) \quad :T' \end{array}\right] \tag{2.1}$$

Provided $k_1, k_2 > 0$

$$\mathrm{Re}\left[\omega + 2k_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right) + 2k_2 \min_{1\le j\le m_1}\left(\frac{d_j}{\delta_j}\right)\right] > 0$$

$$\left|\arg z_1\right| < \frac{A\pi}{2}, \left|\arg z_2\right| < \frac{B\pi}{2}, A, B > 0.$$

The integral (2.1) can be established easily by making use of the definition of I-function of two variables (1.4) and the results (1.5), (1.7) respectively.

3. Solution of the Problem

The solution of the problem to be obtained is

$$y(x,t) = \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty}\left(1-\frac{1}{2\lambda_n\pi}\sin 2\lambda_n\pi\right)^{-1}$$

$$\times I^{m_1, n_1+2:P}_{p+2, q+2:Q}\left[\begin{array}{c|l} z_1 & \left(1-\frac{\omega}{2};k_1,k_2\right), \left(\frac{1-\omega}{2};k_1,k_2\right), \left(e_p:E_p, E'_p\right):T \\ z_2 & \left(f_q;F_q,F'_q\right), \quad \left(\frac{1-\omega\pm\lambda_n}{2}:k_1,k_2\right) \quad :T' \end{array}\right]$$

$$\times \sin\frac{\lambda_n \pi x}{L} \sin\frac{\lambda_n \pi \mu t}{L} \tag{3.1}$$

where all conditions of convergence are same as in (2.1).

Proof : If $t = 0$, then by virtue of (1.1.1), we have

$$\left(\sin\frac{\pi x}{L}\right)^{\rho-1} I\left[\begin{matrix} z_1\left(\sin\frac{\pi x}{L}\right)^{2k_1} \\ \\ z_2\left(\sin\frac{\pi x}{L}\right)^{2k_2}\end{matrix}\right] = \sum_{n=1}^{\infty} A_n \sin\frac{\lambda_n \pi x}{L}. \tag{3.2}$$

Multiply both sides of (3.2) by $\sin \lambda_m \pi x/L$ and integrate with respect to x from 0 to L. Now use the integral (2.1) and orthogonal property of sines (1.6), we thus obtain the value of constant A_m.

Substituting the value of A_m in (3.1), we get the desired solution.

(i) For $m_1 = n_1 = p = q = 0$ in the results (2.1) and (3.1), the resulting equations are in the form of product of two I-function of one variable.

(ii) For $r = 2$, the results in (2.1) and (3.1) reduce to the following results involving H-function of two variables.

REFERENCES

1. Churchill, R. V. : Fourier series and boundary value problems, McGraw-Hill Book Company, New York, (1941).
2. Goyal, Anil and Agrawal, R. D. : Integral involving the product of I-function of two variables, *J. M. A. C. T.*, Vol. 28, (1995) 147-155.
3. Gradshteyn, I. S. and Ryzhik, I. M. : Table of Integrals, Series and Products, Academic Press, New York (1980).
4. Saxena, V. P. : Formal solution of certain new pair of dual integral equations invloving H-function, *Proc. Nat. Acad. Sci. India* 52 (A), (1982), 356-375.
5. Sommerfeld A. : Partial Differential Equations in Physics, Acad. Press, N. Y. (1949).

Department of Applied Mathematics
and Computer Science
S. A. T. I., Vidisha (M. P.) India.

(*Received* March 11, 1997)

SH.-20, Mehta Complex
Nehru Nagar, Bhopal (M. P.) India.